AN INTUITIVE GUIDE TO LEBESGUE MEASURE

Michael Keith

The University of Texas at Austin

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supervised by
Kenneth DeMason

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Abstract. In this paper, we lay the foundation for the study of Lebesgue integration. We rely on techniques of analysis to develop the fundamentals of measure theory. We first offer several important definitions and notations that we utilise throughout. Then we transition to the outer measure of a set before defining the Lebesgue measure of any set and sigma algebras. Afterwards, we list and explain several properties of Lebesgue measure that are pivotal in creating a more generalised characterisation of measurable sets. Next, we comment on said characterisation before turning to measurable functions, which are the backbone of Lebesgue integration. Finally, we conclude with a summary of the utility of the material presented in this paper – namely, we briefly discuss the connection to Lebesgue integration.

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1. Fundamental Definitions

To really grasp an intuitive understanding of the Lebesgue Measure and some of its applications, a few fundamental definitions and theorems regarding Real Analysis need to be established. Additionally, these theorems and definitions will assist in elucidating the notation that will be employed henceforth.

Definition 1.1. The complement of a set E is denoted by E^c. Since a set is just a collection of points in a space, its complement is the entire space with all of the elements of E removed. In the case of E being the entire space, E^c is the empty set, $\emptyset$.

Definition 1.2. The set difference of two sets E and A is given by $E - A$. This means take all of the elements shared by E and A and remove them from E. Symbolically, we may write $E - A = E \cap A^c$.

Definition 1.3. For a sequence of sets $\{E_k\}_{k=1}^{\infty}$, $\limsup E_k = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k$ and $\liminf E_k = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k$.

Definition 1.4. For all elements $a \in A$ and $\epsilon > 0$, A is dense in E if there is an $e \in E$ such that $0 < |e - a| < \epsilon$.

For example, consider the rational numbers. They are dense in $\mathbb{R}$ because there is some rational number that is arbitrarily close to any number in $\mathbb{R}$.

Definition 1.5. The interior of a set, E°, is an open set. The set itself is open iff $E = E^\circ$. If E is open, then E^c is closed and vice versa. Both $\emptyset$ and $\mathbb{R}^n$ are open and closed. Some sets are neither open nor closed. Additionally, we denote an open set as G and a closed set as F.

Proposition 1.6. *The union of arbitrarily many open sets is open; the intersection of finitely many open sets is open. Also, the intersection of arbitrarily many closed sets is closed; the union of finitely many closed sets is closed.*

Definition 1.7. We denote the set type G_δ as the intersection of infinitely many open sets G_k and the set type F_σ as the union of infinitely many closed sets F_k. Note that neither of these types of sets need to be open or closed.

Definition 1.8. A set E is compact if there is a finite subcover for every open cover of E. An cover of E is a family of sets that includes E in the union of the sets in the family. A cover is open if all of the sets in the family are open.

Theorem 1.9 (The Heine-Borel Theorem)**.** *A set E is compact in $\mathbb{R}^n$ if and only if it is closed and bounded.*

Notation 1.10. The functions presented will be denoted by $f = f(x)$ (or will take a similar form) and will be real-valued.

With the preliminary definitions and theorems out of the way, we will now begin an explanation of the Lebesgue Outer Measure.

2. The Lebesgue Outer Measure and its Properties

In this section we will introduce the concept of the outer measure of a general subset E of $\mathbb{R}^n$ and some of its properties. The general idea behind the Lebesgue outer measure is that we may use some countable collection of closed intervals, S, in the dimension in which we are working that cover our general set E to approximate the outer measure within an arbitrarily small ϵ. Essentially, what we are calculating is related to the size of our set. To do this, we consider several closed intervals I_k, each of which is denoted in set-builder notation by

$$I = \{x : a_j \leq x_j \leq b_j, j = 1, ..., n\}.$$

We then compute the volumes of each of the I_k as follows

$$v(I) = \prod_{j=1}^{n} (b_j - a_j).$$

By our construction of I, we have a rectangle with side lengths given by $b_j - a_j$. To get the volume we just simply multiply the side lengths together, which is given by the above notation. Now, let S be a countable collection of these intervals I_k. We now take the sum of each of the volumes of the individual intervals I_k, expressed as

$$\sigma(S) = \sum_{I_k \in S} v(I_k).$$

Here, $\sigma(S)$ represents the approximate outer measure of the set E that is covered by S. We then define the Lebesgue outer measure of E, which we denote as $|E|_e$, as follows

$$|E|_e = \inf \sigma(S).$$

We then note that $0 \leq |E|_e \leq \infty$ because volume cannot be less than zero but may be infinite. Some helpful points are as follows:

(1) the empty set, $\emptyset$, has outer measure equal to zero,
(2) the outer measure of a finite interval is its volume ($|I|_e = v(I)$),
(3) to find the outer measure of a single interval, we cover said interval by itself, and
(4) if, for some $I_k \in S$, $a_j = -\infty$ or $b_j = \infty$, then I_k has infinite measure.

Several important theorems arise out of this definition of Lebesgue outer measure:

Theorem 2.1 (Monotonicity). *If $E_1 \subset E_2$, then $|E_1|_e \leq |E_2|_e$.*

The intuition here is that because $E_1 \subset E_2$, any cover of E_2 must also be a cover of E_1. So, if E_1 is a proper subset of E_2, then we obtain the less than inequality. On the other hand, if $E_1 = E_2$ then the Lebesgue outer measures are equal. We establish monotonicity.

Theorem 2.2 (Subadditivity). *If $E = \bigcup E_k$ is a countable union of sets, then $|E|_e \leq \sum |E_k|_e$.*

If we just consider disjoint E_k, then we are able to add up the individual outer measures of each of the E_k, which equals the outer measure of the set E. On the other hand, if any of the E_k overlap any number of times, the measures of each E_k when summed will include the measures of the overlapping regions more than once. Thus, we must have the inequality to account for this case. We now establish subadditivity.

The following theorem allows us to expand our definition of measure from simple intervals to more generalised open sets.

Theorem 2.3. *If $E \subset \mathbb{R}^n$, then for some $\epsilon > 0$, there is an open set G such that $E \subset G$ and $|G|_e \leq |E|_e + \epsilon$. Hence, $|E|_e = \inf |G|_e$, where the infimum is taken over all open sets G containing E.*

The main idea here is that we may take a collection of intervals that cover the set E and form our desired open set using these intervals. However, a couple of things are happening here. The first is that we are saying we can estimate the outer measure of our set E using a collection of intervals within $\epsilon/2$ error. We have defined $|E|_e$ to be $\inf \sigma(S)$. In doing this, we are taking the infimum of the $\sigma(S)$ across all of the S. If we just let $G = \bigcup I_k$, then, from the previous theorem, $|G|_e \leq \sum |I_k| = \sigma(S) \leq |E|_e + \epsilon/2$. This is great except that G is no longer open. So, we must now, for each interval $I_k \in S$, consider their interiors because we desire I_k to be open. Although we now employ the use of the interiors, these interior may not be enough to cover our set E. Thus, we need to increase the size of each of the I_k so that the union of their interiors, $\bigcup I_k^\circ$, contains E. Yet, we need also adjust the size of each interval by $\epsilon 2^{-k-1}$. Thus, we now have a new collection of intervals I_k' with interiors $(I_k')^\circ$, the union of which is the open set G we need.

This theorem is the beginning of a much more generalised characterisation of the outer measure of some set E, whose result is the foundation for the following theorem.

Theorem 2.4. *If $E \subset \mathbb{R}^n$, there exists a set H of type G_δ such that $E \subset H$ and $|E|_e = |H|_e$.*

This theorem is important because it, along with the aid of Theorem 2.3, establishes the fact that we may take any generalised set in $\mathbb{R}^n$ and contain it within a more simple set, H of type G_δ, that has the same Lebesgue outer measure.

In the following theorem, we denote a rotation of some interval or set by using an apostrophe.

Theorem 2.5. *For every set $E \subset \mathbb{R}^n$, $|E|_e' = |E|_e$.*

The essential concept of this theorem is that we may take a set defined in some space with now rotated coordinate axes, rotate the set correspondingly, and retain the same the Lebesgue outer measure as before the rotation. In the preceding theorems, our intervals of consideration were based on the standard coordinate axes. However, we may elect to operate on a different basis, which requires the use of intervals different than those typically utilised on the standard coordinate axes. Nevertheless, because measure is independent of of the chosen basis – namely, the intervals of concern – our measure remains unchanged despite the rotation. Though the underlying work here is a bit more tedious, it summarises primarily to the generation of rotated intervals, I_k', that perform the same functions as the intervals I_k when calculating the Lebesgue outer measure of E. In other words, we may take the infimum of the sum of the volumes of these I_k' to get $|E|_e'$.

With the fundamental theorems and understandings of Lebesgue outer measure elucidated, we will now transition to defining Lebesgue measure and the definition and construction of sigma algebras.

3. The Lebesgue Measure and Sigma Algebras

Definition 3.1. In general, a set is *Lebesgue measurable* if, for some $\epsilon > 0$, there exists and open set G such that $E \subset G$ and $|G - E|_e < \epsilon$, where $E \subset \mathbb{R}^n$ is the set we desire to measure. If E is measurable, we may then define $|E| = |E|_e$, the Lebesgue measure. Two important consequences that characterise the Lebesgue measure are listed below.

(1) All open sets are measurable.
(2) Sets with outer measure 0 are measurable.

Additionally, several properties of the outer measure are applicable to Lebesgue measure, including monotonicity and subadditivity. While monotonicity is immediate, subadditivity needs to be proved, which we now perform below.

Theorem 3.2. *The union of a countable number of measurable sets is measurable. In addition, if $E = \bigcup E_k$, then $|E| \leq \sum |E_k|$.*

We desire to find an open set G such that $|G - E|_e < \epsilon$. So, the underlying idea is that, for each k, we may find some open set G_k that contains E_k and $|G_k - E_k|_e < \epsilon 2^{-k-1}$. Let $G = \bigcup G_k$ and observe that $G \setminus E \subset \bigcup G_k - E_k$. We may then take the outer measure of $\bigcup (G_k - E_k)$ and apply monotonicity of outer measures. Under the definition of measurability above, we have that E is measurable. Really, this theorem is similar to the subadditivity proof discussed in the preceding section using intervals, though here we use several G_k. Thus, we get an analogous theorem about subadditivity, which applies the definition of Lebesgue measure – the Lebesgue Measure is equal to the Lebesgue Outer Measure.

A corollary that is derived from this theorem is as follows:

Corollary 3.3. *An interval I is measurable and its measure is given as follows: $|I| = v(I)$, where $v(I)$ is the volume of the interval.*

Before we introduce the next theorem, two lemmas need be established. Be warned, their proofs are lengthy and extensive but the ideas behind them are straightforward and rely on the theorems from the previous section and the above corollary.

Lemma 3.4. *If $\{I_k\}_{k=1}^N$ is a finite collection of non-overlapping intervals, then $\bigcup I_k$ is measurable and $|\bigcup I_k| = \sum |I_k|$.*

The proof makes use of Corollary 3.3, the fact that $|I_k| = v(I_k)$, Theorem 3.2, and the Heine-Borel Theorem (the intervals are compact). Yet, the general understanding here should be clear: we should be able to take the measures of any finite number of non-overlapping intervals and add them together to get the measure of the union of these intervals.

Recall here that $d(E_1, E_2) = \inf \{|x_1 - x_2| : x_1 \in E_1, x_2 \in E_2\}$, where d is the distance between the two sets.

Lemma 3.5. *If $d(E_1, E_2) > 0$, then $|E_1 \cup E_2|_e = |E_1|_e + |E_2|_e$.*

Again, the proof is complicated, but we consider a sequence of intervals that cover the union of the two sets E_1 and E_2. As long as each diameter of the I_k is less than the distance between the two sets, we may split the sequence of intervals into two separate sequences that cover E_1 and E_2 separately. In doing this, we may say that $|E_1|_e + |E_2|_e \leq \sum |I_k| \leq |E_1 \cup E_2|_e + \epsilon$. Yet, we may also say that $|E_1 \cup E_2|_e \leq |E_1|_e + |E_2|_e$, implying that $|E_1 \cup E_2|_e = |E_1|_e + |E_2|_e$. This almost gives us the theorem. There is a technical case to consider: the diameter of each of the I_k may be greater than or equal to $d(E_1, E_2)$. In this case, we can partition each of the I_k into finitely many non-overlapping subintervals with diameter less than the distance between the sets of consideration. We then apply the previous lemma.

With these lemmata, we are able to introduce the following theorem:

Theorem 3.6. *Every closed set F is measurable.*

Consider a closed set F that is compact. We can then find some open set G containing F that is close in measure to $|F|_e$ and use the set difference $G - F$ to conclude that F is measurable. We do this by first noting that the set difference $G - F$ is open, so it is the union of non-overlapping closed intervals I_k. Then F is measurable if $|G - F|_e < \epsilon$. Yet, $G - F = \bigcup I_k$, which, by subadditivity, gives that $|G - F|_e \leq \sum |I_k|$. Now, notice that we can write that $G = F \cup (\bigcup I_k)$. Hence, for any N, $F \cup (\bigcup_{k=1}^{N} I_k) \subset G$. By monotonicity and Lemma 3.4 we have $|F|_e + |\bigcup_{k=1}^{N} I_k| \leq |G|$. Then, by Lemma 3.5, $|\bigcup_{k=1}^{N} I_k| = \sum_{k=1}^{N} |I_k|$. Thus, $\sum_{k=1}^{N} |I_k| \leq |G| - |F|_e$ for all N. This is possible because F is compact – that is, $|F|_e$ is finite. However, recall that we chose G such that $|G| - |F|_e < \epsilon$. Thus, for all N, $\sum_{k=1}^{N} |I_k| < \epsilon$, which implies that $\sum |I_k| < \epsilon$. Therefore, $|G - F|_e < \epsilon$ as desired. However, this only works in the case that F is a compact set. Nevertheless, we may intersect F with some of closed and bounded ball of radius k, B_k, to get a compact set, which we may use the above logic to deduce its measurability.

Theorem 3.7. *The complement of a measurable set is measurable.*

This theorem makes use of the fact that closed sets are measurable. Instead of approximating the measure of a set E from the outside using open sets G_k, we may approximate from the inside of E^c by using the complements of G_k, which are closed. Using the preceding theorem we then have that E^c is measurable.

Theorem 3.8. *The intersection of measurable sets is measurable.*

This follows nicely from Theorem 3.2 and Theorem 3.7. Observe that $\bigcap_{n=1}^{\infty} E_k = \bigcup_{n=1}^{\infty} E_k^c$, where E^c is the complement of E. Then we simply have the union of the complements of the E_k, which we know is measurable because unions and complements of measurable sets are measurable.

Theorem 3.9. *For some measurable $E, F \subset \mathbb{R}^n$, $E - F$ is measurable.*

This also follows nicely from Theorem 3.7 since $E - F = E \cap F^c$. We now have an intersection of two sets, with one of them being a complement. Clearly, this set difference is measurable.

We now define a σ-algebra as a collection of subsets of the space we are working in. An example is our frequently used space $\mathbb{R}^n$. Two important properties of σ-algebras are that each σ-algebra is closed under complement and under unions. In fact, they are also closed under intersections! The Lebesgue σ-algebra is defined as the collection of measurable subsets of $\mathbb{R}^n$. It is important to note that we denote the smallest σ-algebra that contains all of the open subsets of $\mathbb{R}^n$ as the Borel σ-algebra.

Theorem 3.10. *Every Borel set is measurable.*

This theorem is very straightforward because the Borel σ-algebra is the smallest σ-algebra contained in the Lebesgue σ-algebra. Since the Lebesgue σ-algebra itself is a σ-algebra that contains all of the open intervals of $\mathbb{R}^n$, we have that the Borel sets are measurable by this containment property.

4. Properties of the Lebesgue Measure

We now make the transition from the definition of Lebesgue Measure, its important theorems, and sigma algebras to the properties of Lebesgue Measure. In the previous sections, we defined a measurable set by approximation from the outside using open sets. Naturally, the question may arise: is this method of approximation required? No! In fact, we may approximate from the inside of a set by using closed sets as the following lemma demonstrates.

Lemma 4.1. *A set $E \in \mathbb{R}^n$ is measurable if and only if, given any $\epsilon > 0$, there exists a closed set $F \subset E$ such that $|E - F|_e < \epsilon$.*

Because the Lebesgue σ-algebra is closed under complement, we are able to take a set E and deduce its measurability by considering its complement, E^c. Now, E^c is measurable if and only if there is some open set G where $E^c \subset G$ and $|G - E|_e < \epsilon$. We also know that G exists if and only if there exists some closed F such that $F = G^c$ – this is true because the complement of an open set is a closed set. Then $F \subset E$ with $|E - F|_e < \epsilon$ since $G - E^c = E - F$.

Theorem 4.2. *If $\{E_k\}$ is a countable collection of disjoint measurable sets, then $|\bigcup_k E_k| = \sum_k |E_k|$.*

Proof. Let $\epsilon > 0$ and assume each of the E_k are bounded. We are able to use the previous lemma, for each k, to choose a closed set $F_k \subset E_k$ so that $|E_k - F_k| < \epsilon 2^{-k}$. Let $G_1 = E_k - F_k$ and $G_2 = F_k$. Hence, $G_1 \cup G_2 = E_k$ and $|G_1 \cup G_2| \leq |G_1| + |G_2|$. Then $|E_k| \leq |F_k| + \epsilon 2^{-k}$. Since we assume the E_k are bounded and disjoint, the F_k are compact and disjoint (note: the F_k are closed and bounded, hence compact). By Lemma 3.5, $|\bigcup_{k=1}^m F_k| = \sum_{k=1}^m |F_k|$. Then since $\bigcup_{k=1}^m F_k \subset \bigcup_{k=1}^\infty F_k \subset \bigcup_k E_k$, it follows that $\sum_{k=1}^m |F_k| \leq |\bigcup_k E_k|$. In particular, this is true for every $m \geq 1$ and in the limit. We then obtain $|\bigcup_k E_k| \geq \sum_k |F_k| \geq \sum_k (|E_k| - \epsilon 2^{-k}) = \sum_k |E_k| - \epsilon$. We may then have the reverse inequality by Theorem 3.2. □

Corollary 4.3. *If $\{I_k\}$ is a sequence of non-overlapping intervals, then $|\bigcup I_k| = \sum |I_k|$.*

The idea behind this corollary is that we can consider the interiors, I_k°, of each of the I_k, which are disjoint. We first have that $|\bigcup I_k| \leq \sum |I_k|$. On the other hand, $|\bigcup I_k^\circ| = \sum |I_k^\circ|$, since the I_k° are disjoint. Yet, $|\bigcup I_k| \geq |\bigcup I_k^\circ|$. We already know that $\sum |I_k^\circ| = \sum |I_k|$ because $|I_k^\circ| = |I_k|$ as these sets differ by a set that has zero measure. Hence, $|\bigcup I_k| \geq \sum |I_k|$. Since we have shown that both inequalities are true, we have the equality.

Corollary 4.4. *Suppose there are two measurable sets E_1 and E_2 with $E_2 \subset E_1$ and $|E_2|$ finite. Then $|E_1 - E_2| = |E_1| - |E_2|$.*

Proof. Notice that since $E_2 \subset E_1$ we have $(E_1 - E_2) \cup E_2 = E_1$. This union is disjoint, so by Theorem 4.2 we have that $|E_1 - E_2| + |E_2| = |E_1|$. Yet, $|E_2|$ is finite, so $|E_1 - E_2| = |E_1| - |E_2|$. □

Theorem 4.5. *Let $\{E_k\}_{k=1}^\infty$ be a collection of measurable sets. Then*

(1) *If E_k increases to E, then $\lim_{k\to\infty} |E_k| = |E|$; and*
(2) *If E_k decreases to E and $|E_k|$ is finite for some k, then $\lim_{k\to\infty} |E_k| = |E|$.*

In either case, we write E such that it is the union of disjoint measurable sets. For example, in the first case we write E as

$$E = E_1 \cup (E_2 - E_1) \cup (E_3 - E_2) \cup ...$$

By this construction, we may use Theorem 4.2 and Corollary 4.4 to break apart the measures as sums and differences. However, note that the the constructions of E slightly differ in each case but we are able to derive the same result. Nevertheless, it is crucial that the $|E_k|$ be finite in the second case. Why? Consider a sequence of sets $\{E_k\}_{k=1}^{\infty}$ such that $|E_k| = \infty$ for all k. One way to construct such a sequence of sets would be to consider $\{(1, \infty), (2, \infty), (3, \infty), ...\}$. Notice that the intersection of the first N many of these sets approaches the empty set, which has measure equal to zero. However, the $\lim_{k\to\infty} |E_k| \neq |E|$ because the measures of the sets themselves clearly are infinite. Thus, the $|E_k|$ need be finite.

5. Measurable Sets and their Character

We previously employed open sets to approximate the measure of a general set E. However, Lemma 4.1 shows how we may even use closed sets to approximate the measure of a set E. Better yet, we found that we could state whether a set was measurable by being able to find closed or open sets F, G either contained within E or containing E. The question then arises: are the only ways to characterise measurable sets by using open and closed sets? The answer is no. There are other ways in which we may conclude the measurability of a set, and the following theorems expound such methods.

Theorem 5.1.

(1) E is measurable if and only if $E = H - Z$, H is of type G_δ, and $|Z| = 0$.
(2) E is measurable if and only if $E = H \cup Z$, H is of type F_σ, and $|Z| = 0$.

If we can write E in either of these ways, then it is measurable since H and Z are measurable in either case by the theorems from the former sections. However, we now need to prove that E being measurable implies that we can find some such sets H and Z such that $E = H - Z$ or $E = H \cup Z$. In the first case, suppose E is measurable. Then, for each k, we may choose a sequence of sets G_k such that $E \subset G_k$ and $|G_k - E| < 1/k$. We can then take the intersection of the G_k, which we will denote as H. Then E is contained in H, and H is of type G_δ. Since $|G_k - E| < 1/k$ and $|H - E| \leq |G_k - E|$ for all k, it follows that $|H - E| = 0$. In the second case, we need only assume E is measurable and then we know that its complement, E^c, is measurable. We can then apply the first case to E^c. That is, write E^c as the set difference of the intersection, for each k, of the G_k and Z, that is, $E^c = \bigcap G_k - Z$, where Z is a set of zero measure. Then, assuming $|Z| = 0$, $E = \bigcup G_k^c \cup Z$. Let $H = \bigcup G_k^c$, which is of type F_σ. We then have the result.

This next theorem, though loosely stated, offers us profound illumination on the structure of measurable sets.

Theorem 5.2 (Littlewood's First Principle of Analysis)**.** *Any measurable subset of $\mathbb{R}$ is essentially a finite union of non-overlapping intervals.*

This next theorem is likely the most important characterisation of measurability because it allows us to broaden measurability to spaces that may not behave like $\mathbb{R}^n$.

Theorem 5.3 (Carathéodory's Criterion for Measurability)**.** *E is measurable if and only if for every A, $|A|_e = |A \cap E|_e + |A - E|_e$.*

Make note that in this theorem, A need not be a measurable set. We consider two cases: $|E| < \infty$ and $|E| = \infty$. In the first case, we assume that E is measurable and suppose, for some A, that $A \subset H$, where $H = (H \cap E) \cup (H - E)$ is of type G_δ. We may find such an H given A by Theorem 2.4. Then $|A|_e = |H|$. In choosing H so that $|H| = |A|_e$, we have that $|A|_e = |H \cap E| + |H \setminus E|$. We then know that this is greater than or equal to $|A \cap E|_e + |A - E|_e$ by monotonicity. However, the reverse inequality, $|A|_e \leq |A \cap E|_e + |A - E|_e$, is immediately true (subadditivity), so we have equality. We now will show that E is measurable given that $|A|_e = |A \cap E|_e + |A - E|_e$ for all $A \subset \mathbb{R}^n$. This is true if $|E|_e < \infty$ because we can find another set of type $G_\delta = H$ such that $E \subset H$ and $|E|_e = |H|$. Write H as $H = E \cup (H - E)$. Then $|H| = |E|_e + |H - E|_e$ as we have the union of disjoint measurable sets. By our choice of H, we know that $|H - E|_e = 0$. Thus, E must be measurable because $H - E$ is a set with measure zero. However, what if E has infinite outer measure? In this case, we construct an increasing, finite sequence of sets that converge to E. We can then apply the logic from above – namely, the logic of the finite case – to each of the sets within the sequence. Thus, when we take the limit, the proof still works.

Corollary 5.4. *If E is a measurable subset of A, then $|A|_e = |E| + |A - E|_e$.*

This follows directly from the theorem above. Since $E \subset A$, then $(A \cap E) = E$. Also, as long as $|E|$ is finite, then we have that $|A|_e - |E| = |A - E|_e$.

6. Measurable Functions

We now transition from measurable sets to measurable functions. Measurable functions are necessary for Lebesgue integration and in probability theory where such functions are at the center of the calculation of probabilities of random variables. The final theorem presented relates measurable functions to simple functions, which is crucial in the development of Lebesgue integration.

Definition 6.1. For a set $E \in \mathbb{R}^n$, a function f being real-valued on E ($-\infty \leq f(x) \leq \infty, x \in E$) is Lebesgue Measurable if, for every a,

$$\{x \in E : f(x) > a\} = \{f > a\}$$

is measurable.

Theorem 6.2. *For a real-valued function f that is measurable on a set E, f is measurable if and only if any of the following*

(1) $\{f \geq a\}$ is measurable
(2) $\{f < a\}$ is measurable
(3) $\{f \leq a\}$ is measurable

are true for all finite a.

For (1), if we assume that f is a measurable function, we get that $\{f > a\}$ is measurable. Since the set in (1) is an infinite intersection of $\{f > a - 1/k\}$, then we have that (1) is measurable. The other two of these rely on measurability being closed under complements. For example, if f is measurable, we know by the above definition that $\{f > a\}$ is measurable. Since the complement of $\{f > a\}$ is $\{f \leq a\}$, then we know (3) is true. Similarly, $\{f < a\}$ is measurable since it is the complement of (1).

Corollary 6.3. *Let f be a function defined on a measurable set E. If f is measurable, then so are $\{f > -\infty\}, \{f < \infty\}, \{f = \infty\}, \{a \leq f \leq b\}, \{f = a\}$, etc...*

Each of these really simplify to taking unions, intersections, and complements of other known sets. Take $\{f < \infty\}$. This is equivalent to $\bigcup_{k=1}^{\infty}\{f < k\}$. We can do something similar for the other sets.

For a function f defined on E, we now denote the preimage of a set P as the function $f^{-1}(P)$. That is, all of the elements $x \in E$ such that $f(x) \in P$.

Theorem 6.4. *Let f be a function defined on a measurable set E. If f is measurable, then for every open set $G \subset \mathbb{R}$, $f^{-1}(G)$ is a measurable subset of $\mathbb{R}^n$. Also, f is measurable if $f^{-1}(G)$ is measurable for all open sets $G \subset \mathbb{R}^n$ and either $\{f = \infty\}$ or $\{f = -\infty\}$.*

Here the idea is that if G is an open set, then it is the union of disjoint open intervals, (a_k, b_k). Then the preimage of any of the intervals under f is the set $\{a_k < f < b_k\}$, which we know to be measurable from the previous corollary. Hence, the preimage of G under f is just the union of all of the preimages of each interval that comprises G. That is, $f^{-1}(G) = \bigcup f^{-1}((a_k, b_k))$, which is just a union of measurable sets.

Theorem 6.5. *For a dense subset $A \subset \mathbb{R}$, f is measurable if, for all $a \in A$, $\{f > a\}$ is measurable.*

We can consider a decreasing sequence $\{a_k\}_{k=1}^{\infty}$ in A that converges to a. The union of the sets $\{f > a_k\}$ is just $\{f > a\}$. Clearly, this is measurable.

Theorem 6.6. *If f is measurable and $g = f$ almost everywhere (a.e.), then g is measurable and $|\{g > a\}| = |\{f > a\}|$.*

We can construct a set from f and g – specifically, $\{f \neq g\}$ – that has measure equal to zero. The union of this set and $\{g > a\}$ is equal to the the union of this zero measure set and $\{f > a\}$. These new sets have a measure that differ by zero. Thus, they are essentially $\{g > a\}$ and $\{f > a\}$, respectively. We can then conclude that g is measurable.

Theorem 6.7. *If g is continuous and f is finite a.e. in E, then $g(f)$ is defined a.e. in E and if f is measurable, then so is $g(f)$.*

We just need to show that the composition of the preimages under these functions is measurable, so this theorem relies on Theorem 6.4. We can consider $f^{-1}(g^{-1}(G)) = \{x : g(f(x)) \in G\}$ using the definition of a preimage. Since we assume f to be measurable and finite in E and $g^{-1}(G)$ is open, then we have that $f^{-1}(g^{-1}(G))$ is measurable.

Theorem 6.8. *If f and g are measurable, then $\{f > g\}$ is measurable.*

Proof. Enumerating the rationals as $\{Q_k\}$, we have

$$\{f > g\} = \bigcup_{k=1}^{\infty} \{f > Q_k > g\} = \bigcup_{k=1}^{\infty} (\{f > Q_k\} \cap \{g < Q_k\}).$$

We are able to do this because the set of rational numbers is countable and dense. Since f, g are measurable, we have that $\{g < Q_k\}, \{f > Q_k\}$ are measurable by Theorem 6.2. Since intersecting these sets preserves measurability, we have the theorem. □

Theorem 6.9. *If f is measurable, then, for any constant $c \in \mathbb{R}$, $f + c$ and cf are measurable.*

Proof. Suppose that $c \in \mathbb{R}$ and f is measurable. Then since $f = \{f > a\}$, $cf = \{cf > a\}$. Notice that $\{cf > a\}$ implies that $\{f < a/c\}$ for $c < 0$ or $\{f > a/c\}$ for $c > 0$. If $c = 0$, then $cf = 0$, which is measurable. Thus, cf is measurable. For $f + c$, we have $\{f + c < a\} = \{f < a - c\}$, where $a - c$ is just some other value in $\mathbb{R}$. Thus, $f + c$ is measurable. □

Theorem 6.10. *If f and g are measurable, then $f + g$ is measurable.*

Proof. We need to show that $\{f + g > a\}$. However, we know that both f and g are measurable. Thus, $\{f + g > a\} = \{f > a - g\}$, which is measurable by the previous two theorems. □

Theorem 6.11. *If f and g are measurable, then fg is measurable; f/g is measurable as long as $g \neq 0$ a.e.*

Proof. We leave the proof of the first part of this theorem – namely, the fact that fg is measurable – to the reader. For the quotient, first suppose that $1/g$, f are measurable and that $g \neq 0$. Then $f/g = (1/g)f$, which is a measurable product of functions. We can say that $1/g$ is measurable because $\{1/g < a\}$ implies any of the following

(1) $\{1/a < g < 0\}$ if $a < 0$,
(2) $\{-\infty < g < 0\}$ if $a = 0$, or
(3) $\{-\infty < g < 0\} \cup \{1/a < g < \infty\}$ if $a > 0$,

which are all measurable sets. □

Theorem 6.12. *For a sequence of measurable functions, $\{f_k(x)\}_{k=1}^{\infty}$, then $\sup_k f_k(x)$ and $\inf_k f(x)$ are measurable.*

Notice that, for $k \geq 1$, $\{\sup_k f_k > a\} = \bigcup_{k=1}^{\infty}\{f_k > a\}$. Clearly, this is measurable. Yet, $\inf_k f_k = -\sup_k(-f_k)$, which is a constant multiple of a measurable function. Thus, $\inf_k f_k$ is measurable.

Theorem 6.13. *If $\{f_k\}$ is a sequence of measurable functions, then $\limsup_{k\to\infty} f_k$ and $\liminf_{k\to\infty} f_k$ are measurable. Also, if the $\lim_{k\to\infty} f_k$ exists a.e., then it, too, is measurable.*

The first statement comes from the previous theorem because we may define, for sequences,

$$\limsup_{k\to\infty} f_k = \inf_j\{\sup f_k\}_{k\geq j}$$

and

$$\liminf_{k\to\infty} f_k = \sup_j\{\inf f_k\}_{k\geq j}.$$

The second statement then comes from the fact that, if the limit exists,

$$\lim_{k\to\infty} f_k = \limsup_{k\to\infty} f_k.$$

Theorem 6.14. *For a function f, we define $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Then, if f is measurable, so are f^+, f^-.*

Proof. Suppose f is measurable. If f^+ exists, then it is the $\sup f$, hence measurable. Similarly, if $\min\{f, 0\}$ exists, then it is the $\inf f$. Thus, f^- is measurable since it is the constant multiple of a measurable function. □

In the next theorem, we begin discussion of simple functions. A simple function is a function f that only takes on finitely many finite values on the set on which it is defined. We prefer working with simple functions because they enable us to work with more complicated functions on a simpler level. This is true since we are able to approximate any function using simple functions.

Theorem 6.15.

(1) Every f can be written as the limit of a sequence $\{f_k\}$ of simple functions.
(2) If $f \geq 0$, then we can choose the sequence of functions such that increase to f.
(3) If f in either of the above two cases is measurable, then f_k, the sequence of measurable functions, can be chosen so that it is measurable.

For this theorem, we first offer an explanation of (2), we then rationalise how (2) gives us (1). Finally, we offer reason as to why (3) is true. For $f \geq 0$, we partition its range into special subintervals that to generate a sequence of simple functions that reside in the domain of f. By the construction of these subintervals, each simple function in the sequence increases to f. In (1), we may break f into f^+ and f^-, each of which are non-negative, and suppose they have increasing sequences of functions. The difference of these sequences is a simple function. We are then able to apply the result from (2) to these sequences of simple functions to get the result. For (3), suppose $f \geq 0$. From (2) we have a sequence of measurable functions. These simple functions are linear combinations of characteristic functions of the preimages of the subintervals. Therefore, if f is a measurable function, then the preimages are measurable sets.

The question arises: what is the utility of all of this theory? As stated previously, all of this is crucial in Lebesgue integration. The purpose of the material prior to the introduction of simple functions is to establish the foundations for measurable functions, which are based on measurable sets. Simple functions allow us to take any function, break it into simpler functions, and then apply the measurability of the simple functions to conclude that any continuous function is measurable. We need this to be able to perform Lebesgue integration – integration that relies on the partitioning of the y-axis – as opposed to the x-axis in Riemann integration – to form rectangles to approximate the area under a curve. What makes this form of integration more difficult is that fact that the inverse image of a function needs to be measured, which relies on measurability of sets.

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